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Solutions of dynamic problems for incompressible and slightly compressible helically orthotropic non-uniform thick-walled cylinders☆

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Abstract

Exact analytic solutions are obtained of the one-dimensional dynamic problem for an incompressible elastic radially non-uniform helically orthotropic thick-walled cylinder under plane-strain conditions, loaded with a time-dependent pressure from inside and/or outside. The necessary and sufficient conditions for solutions to exist and to be unique are established. The convergence of the wave solutions for slightly compressible cylinders to the analytic relations obtained for incompressible cylinders is investigated.

Structures in the form of multilayer cylindrical thick-walled shells, layers of which take the form of helically reinforced composite materials, are widely used in practice. In a number of cases, these structures are deformed elastically until they fracture. Such layers are usually considered in cylindrical coordinates as helically orthotropic elastic media, one of the principal axes of anisotropy of which is always directed along the radial coordinate, while the two others are rotated by a certain angle of reinforcement with respect to the longitudinal axis and the direction of the circumferential coordinate.[1] The angle of reinforcement, and also the density and elastic characteristics of composite materials may be piecewise-continuous functions of the radius. Such structural components are sometimes subjected to axisymmetric dynamic loading by a pressure pulse (an internal explosion etc.). A representation of the unsteady stress-strain state in such structures can often be obtained by one-dimensional dynamic calculations for infinitely long cylinders under plane deformation.[2–4] Similar solutions for compressible materials, even in the case of isotropy, are extremely complicated and are written either in series or in the form of improper complex integrals, containing special cylindrical functions.[4] If the compressibility of the layers can be neglected, then, as will be shown below, the solutions can be written in quadratures in the general case, while for many special forms of loading and radial non-uniformity, which are of practical importance, they can be written in terms of elementary functions.

Exact one-dimensional analytical solutions for multilayer elastic hollow cylinders, made of isotropic incompressible materials and loaded with a time-dependent pressure, were obtained in Ref. 3 by developing the approach used by Agababyan.[2] Here these results are extended to the case of layers with helical orthotropy and an arbitrary radial non-uniformity.

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Consider an incompressible infinitely long thick-walled uniform or non-uniform (in particular, multilayer) cylinder under plane strain conditions, loaded with a pulsed axisymmetric pressure $P_1(t)$ on the inner surface and $P_2(t)$ on the outer surface. At the initial instant of time $t=0$ the cylinder is stress-free and fixed, and contact between the layers in the case of a multilayer cylinder is assumed to be ideal. The equation of motion in cylindrical coordinates, taking the axial symmetry and plane strain into account, has the form

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\varphi}{r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where ρ is the density of the material at the point considered, u is the radial displacement, and σ_r and σ_φ are the components of the stress tensor.

The following geometrical Cauchy relations hold for the components of the stress tensor

$$\varepsilon_r = \partial u/\partial r, \quad \varepsilon_\varphi = u/r, \quad \varepsilon_x = \gamma_{xr} = \gamma_{x\varphi} = \gamma_{r\varphi} = 0 \tag{2}$$

There are no shear stresses τ_{xr} and $\tau_{r\varphi}$. The remaining components of the stress and strain tensors are related by physical equations of the theory of elasticity for a helically orthotropic material, which, in vector form, can be written as[5]

$$\{\varepsilon_x, \varepsilon_\varphi, \varepsilon_r, \gamma_{x\varphi}\} = C\{\sigma_x, \sigma_\varphi, \sigma_r, \tau_{x\varphi}\} \tag{3}$$

The 4×4 square matrix C is symmetrical ($c_{ij}=c_{ji}$) and its elements are as follows:

$$\begin{aligned}
c_{11} &= \frac{\cos^4\alpha}{E_x}+\left(\frac{1}{4G_{x\varphi}}-\frac{\nu_{x\varphi}}{2E_x}\right)\sin^2 2\alpha+\frac{\sin^4\alpha}{E_\varphi}\\
c_{12} &= \left(\frac{1}{E_x}+\frac{1}{E_\varphi}+\frac{2\nu_{x\varphi}}{E_x}-\frac{1}{G_{x\varphi}}\right)\frac{\sin^2 2\alpha}{4}-\frac{\nu_{x\varphi}}{E_x}\\
c_{13} &= -\frac{\nu_{r\varphi}\sin^2\alpha+\nu_{rx}\cos^2\alpha}{E_r}, \quad c_{23} = -\frac{\nu_{r\varphi}\cos^2\alpha+\nu_{rx}\sin^2\alpha}{E_r}\\
c_{14} &= \left[2\left(\frac{\sin^2\alpha}{E_\varphi}-\frac{\cos^2\alpha}{E_x}\right)+\left(\frac{1}{G_{x\varphi}}-\frac{2\nu_{x\varphi}}{E_x}\right)\cos 2\alpha\right]\frac{\sin 2\alpha}{2}\\
c_{22} &= \frac{\sin^4\alpha}{E_x}+\left(\frac{1}{4G_{x\varphi}}-\frac{\nu_{x\varphi}}{2E_x}\right)\sin^2 2\alpha+\frac{\cos^4\alpha}{E_\varphi}\\
c_{24} &= \left[2\left(\frac{\cos^2\alpha}{E_\varphi}-\frac{\sin^2\alpha}{E_x}\right)-\left(\frac{1}{G_{x\varphi}}-\frac{2\nu_{x\varphi}}{E_x}\right)\cos 2\alpha\right]\frac{\sin 2\alpha}{2}\\
c_{33} &= \frac{1}{E_r}, \quad c_{34} = \frac{\nu_{rx}-\nu_{r\varphi}}{E_r}\sin 2\alpha, \quad c_{44} = \frac{\cos^2 2\alpha}{G_{x\varphi}}+\left(\frac{1}{E_x}+\frac{1}{E_\varphi}+\frac{2\nu_{x\varphi}}{E_x}\right)\sin^2 2\alpha
\end{aligned} \tag{4}$$

where E_i, G_{ij}, $\nu_{ij}(i, j=x, \varphi, r;\ i\neq j)$ are the technical characteristics of elasticity of the orthotropic material in the principal axes of anisotropy for zero angle of reinforcement ($\alpha=0$), i.e. in the case of cylindrical orthotropy.

The following three equations also hold for any orthotropic medium[5]

$$E_i\nu_{ji} = E_j\nu_{ij} \tag{5}$$

and hence the number of independent elastic characteristics is equal to 9. For incompressible materials, in addition to Eq. (5), the following additional three equations must be satisfied

$$\nu_{ij}+\nu_{ik} = 1; \quad i\neq j\neq k\neq i \tag{6}$$

Solving system (5), (6) for ν_{ij}, we obtain

$$\nu_{ij} = \frac{1}{2} + \frac{E_i}{2E_j} - \frac{E_i^2 E_j}{2E_r E_x E_\varphi} \tag{7}$$

Hence, whereas in a compressible orthotropic material there are 9 independent elasticity characteristics in all,[5] in an incompressible material there are only 6. If we take as the main (basic) characteristics Young's modulus and the shear modulus in the appropriate principal directions, all the transverse strain (Poisson) ratios will be uniquely defined by formulae (7).

It can also be shown that, for an incompressible helically orthotropic material, the following four equations for the elements of the compliance matrix (4) are satisfied identically with respect to α

$$c_{m1} + c_{m2} + c_{m3} = 0; \quad m = 1, 2, 3, 4 \tag{8}$$

Hooke's law (3) for plane strain (2) and an incompressible helically orthotropic material (4)–(8) can be converted to the form

$$\begin{aligned} &\varepsilon_r + \varepsilon_\varphi = 0, \quad B(\varepsilon_r - \varepsilon_\varphi) = 2A(\sigma_r - \sigma_\varphi) \\ &B\sigma_x = \sigma_\varphi(c_{14}c_{24} - c_{44}c_{12}) + \sigma_r(c_{14}c_{34} - c_{44}c_{13}) \\ &B\tau_{x\varphi} = [c_{33}c_{24} - c_{22}c_{34} + c_{23}(c_{24} - c_{34})](\sigma_r - \sigma_\varphi) \end{aligned} \tag{9}$$

where

$$B = c_{44}c_{11} - c_{14}^2, \quad A = c_{44}(c_{22}c_{33} - c_{23}^2) + 2c_{23}c_{24}c_{34} - c_{22}c_{34}^2 - c_{33}c_{24}^2 \tag{10}$$

and all the ν_{ij} must be defined as in (7).

Substituting expressions (2) into the first equation of (9), we obtain an equation for the bending, by solving which we find

$$u(r, t) = Y(t)/r \tag{11}$$

We will first consider the case when the following condition is satisfied for all r

$$AB > 0 \tag{12}$$

As will be shown below, the cases $AB \le 0$ have no physical meaning and lead to paradoxical or ambiguous solutions.

When condition (12) is satisfied, it follows from relations (2), (11) and the second equation of (9) that

$$\sigma_r - \sigma_\varphi = -BY(t)/(Ar^2) \tag{13}$$

Substituting expressions (11) and (13) into the equation of motion (1), we have

$$\frac{\partial \sigma_r}{\partial r} = \rho\frac{Y''(t)}{r} + \frac{BY(t)}{Ar^3} \tag{14}$$

Integrating this equation with respect to r from the inner radius R_1 to the outer radius R_2 of the cylinder, taking into account the boundary conditions

$$\sigma_r(R_l, t) = -P_l(t), \quad l = 1, 2 \tag{15}$$

we obtain an ordinary differential equation for the function $Y(t)$

$$Y'' + \omega^2 Y = F(t) \tag{16}$$

where

$$F(t) = \frac{P_1(t) - P_2(t)}{M}, \quad M = \int_{R_1}^{R_2} \frac{\rho}{r} dr, \quad \omega^2 = \frac{1}{M} \int_{R_1}^{R_2} \frac{B}{Ar^3} dr \tag{17}$$

The quantities ρ, A and B are introduced under the integral sign, since they may depend on the actual radius.

Integrating Eq. (16) with zero initial conditions $Y(0) = Y'(0) = 0$, we obtain[6]

$$Y(t) = \frac{1}{\omega} \int_0^t F(\xi) \sin \omega (t - \xi) d\xi \tag{18}$$

Knowing $Y(t)$, the stresses can be found in the following way. It follows from Eqs. (14) and (16) that

$$\frac{\partial \sigma_r}{\partial r} = (F - \omega^2 Y) \frac{\rho}{r} + \frac{BY}{Ar^3} \tag{19}$$

Integrating, we obtain

$$\sigma_r(r, t) = -P_1(t) + [F(t) - \omega^2 Y(t)] \int_{R_1}^{r} \frac{\rho}{r} dr + Y(t) \int_{R_1}^{r} \frac{B}{Ar^3} dr \tag{20}$$

Now the stress σ_φ is found from relation (13), while σ_x and $\tau_{x\varphi}$ are found from the last two equations of (9). The strains are found from relations (2) and (11): $\varepsilon_\varphi = -\varepsilon_r = Y(t)/r^2$.

Hence, the problem has been solved for case (12). For a quite wide class of functions of the loads $F(t)$, the time integral (18) can be written in terms of elementary functions.[6] The spatial integrals in formulae (17) and (20) can also be evaluated accurately in a large number of situations, for example, for a multilayer cylinder, when ρ, A and B are piecewise-constant along the radial coordinate.

We will further consider the case when

$$AB \le 0 \tag{21}$$

using the example of a uniform hollow incompressible cylinder (the density, the reinforcement angle and the constants of elasticity are independent of r). To fix our ideas, the loads $P_1(t)$ and $P_2(t)$ will be assumed to be such that

$$F(t) \ge 0, \quad 0 < \int_0^\infty F(t) dt < \infty$$

As was pointed out above, the case (21) has no physical meaning. If we impose additional limitations on the moduli of elasticity E_x, E_r, E_φ and the shear modulus $G_{x\varphi}$, in addition to the requirements that they must be positive and bounded, we can rigorously show that the following 7 cases are possible in an incompressible cylinder when condition (21) is satisfied.

1°. $AB < 0$. The strains and displacements increase without limit with time, and the stresses are determined uniquely (an example is a cylindrically orthotropic material $E_r/E_x = E_\varphi/E_x = 5$).

2°. $B \ne 0, A = 0$. The strains and displacements are identically equal to zero, and all the components of the stress tensor are defined non-uniquely (an example is a cylindrically orthotropic material, $E_r/E_x = E_\varphi/E_x = 4$).

3°. $B = 0$, $c_{14} \ne 0$, $c_{44}c_{12} \ne c_{14}c_{24}$. The strains and displacements increase without limit with time, and the stresses are defined uniquely (an example is a helically orthotropic material, $\alpha = \pi/4$, $E_x/E_\varphi = 2$, $E_r/E_\varphi = 12$, $G_{x\varphi}/E_\varphi = 420$).

4°. $B=0, c_{14} \neq 0, c_{44}c_{12}=c_{14}c_{24}$. The strains and displacements are bounded, and the stresses are defined non-uniquely – there is an innumerable set of solutions for σ_x and $\tau_{\varphi x}$ (an example is a helically orthotropic material, $\alpha=\pi/6$, $E_x/E_r=16$, $E_x/E_\varphi=9$).
5°. $B=c_{14}=c_{44}=0,\ c_{13} \neq 0$. The strains and displacements increase without limit with time, and the stresses are defined uniquely (an example is a helically orthotropic material, $\alpha=\pi/6$, $E_\varphi/E_x=2$, $E_\varphi/E_r=6.5$, $E_\varphi/G_{x\varphi}=1.5$).
6°. $B=c_{14}=c_{44}=c_{13}=0$. The strains and displacements are bounded, and the stresses are defined non-uniquely – there is an innumerable set of solutions for $\tau_{\varphi x}$ (an example is a helically orthotropic material, $\alpha=\pi/4$, $E_\varphi/E_r=2$, $E_x/E_r=4$).
7°. $B=c_{14}=c_{11}=0$. The strains and displacements increase without limit with time, and the stresses are defined uniquely (an example is a helically orthotropic material, $E_\varphi/E_x=\mathrm{tg}^4\alpha$, $E_\varphi(1/G_{x\varphi}+1/E_r)=\cos^2 2\alpha/\cos^4\alpha$; $\alpha \neq \pi n/4$, $n \in \mathrm{Z}$.

These examples of hypothetical materials, for which cases 1°–7° occur, are not unique. Details of the proofs of propositions 1°–7° are omitted here.

Hence, one can only model a helically orthotropic material in problems with cylindrical symmetry by an incompressible elastic material when condition (12) is satisfied. If this requirement is not satisfied, such modelling may lead either to unstable or non-unique solutions.

The paradoxes in cases 1°–7° arise due to singularities, related to the incompressibility of an elastic anisotropic material. The classical theory of elasticity of an anisotropic medium[5] is constructed on the assumption that the specific potential energy of elastic deformation W is positive-definite. If the material is incompressible, this postulate breaks down: one can always choose a load such that the strain will be identically equal to zero for non-zero stresses, i.e. $W \equiv 0$, when not all σ_i and τ_{ij} are equal to zero. In the case of extremely anisotropic incompressible materials, the situation may change even more sharply: as a consequence of Eq. (7) some ν_{ij} may turn out to be negative, and others may turn out to be greater than unity. Because of this, W may also take negative values. If the work of the external forces is finite, i.e. the energy obtained by the system from outside during the time and as a result of the action of the external load, the elastic potential energy of the cylinder Π will be negative, while the kinetic energy K will be positive. The energy balance will then be satisfied; together with the fact that $\Pi \to -\infty$, $K \to +\infty$ as $t \to \infty$, cases 1°, 3°, 5° and 7° will correspond to this. In the remaining cases (2°, 4°, 6° and also in the case when condition (12) is satisfied), the specific potential energy W may vanish for certain non-zero stress tensors, and the strains and displacements will then be unique and bounded, while a unique solution for the stresses is only obtained in the case when condition (12) is satisfied.

Hence, for cylinders with helical anisotropy, a necessary condition for obtaining adequate solutions using the model of an incompressible material is strict inequality (12). By taking the limit it can be shown that relation (12) is also the sufficient condition.

We will briefly note a way of proving this assertion using the example of an isotropic uniform cylinder. The calculations are more complicated for multilayer anisotropic cylinders, but the gist of the proof is essentially unchanged.

We will formulate the boundary-value problem for an isotropic slightly compressible cylinder in the variables

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}; \quad a = \frac{\Omega}{\sqrt{\delta}}, \quad \Omega = \sqrt{\frac{G}{\rho}\left(\delta+\frac{1}{2}\right)}, \quad \delta = \frac{1}{2}-\nu > 0 \tag{22}$$

$$t = 0: u = \frac{\partial u}{\partial t} = 0; \quad r = R_l: \quad \frac{\partial u}{\partial r}-\frac{u}{r}+\frac{1}{2\delta}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right) = -\frac{P_l(t)}{G}, \quad l = 1, 2 \tag{23}$$

where a is the velocity of sound, ν is Poisson's ratio and G is the shear modulus. It can be verified that requirement (12) is satisfied for this problem.

We will solve boundary-value problem (22), (23) by using a Laplace integral transformation with respect to time[7]

$$u(r,t) \leftrightarrow U(r,s), \quad P_l(t) \leftrightarrow \Phi_l(s), \quad l = 1, 2$$

The general solution of the equation of motion (22) in transforms is

$$U(r,s) = D_I(s)I_1(sr/a) + D_K(s)K_1(sr/a) \tag{24}$$

where $I_n(z)$ is the modified n-th order Bessel function and $K_n(z)$ is the n-th order MacDonald function.[8] To determine the constants of integration $D_I(s)$ and $D_K(s)$, we will write the boundary conditions for $r = R_l$ (23) in terms of transforms. Taking the well-known properties of cylindrical functions and their derivatives into account,[8] we will have

$$\begin{aligned} -\frac{\Phi_l(s)}{G} &= D_I(s)\left[\frac{s}{2a\delta}(1+2\delta)I_0\left(\frac{sR_l}{a}\right) - \frac{2}{R_l}I_1\left(\frac{sR_l}{a}\right)\right] - \\ &- D_K(s)\left[\frac{s}{2a\delta}(1+2\delta)K_0\left(\frac{sR_l}{a}\right) + \frac{2}{R_l}K_1\left(\frac{sR_l}{a}\right)\right], \quad l = 1, 2 \end{aligned} \tag{25}$$

When $\delta \to +0$, the arguments of the cylindrical functions in relations (24) and (25) will also be infinitesimals of the order of $\sqrt{\delta}$. Using the well-known expansions of cylindrical functions in the neighbourhood of zero[8] and solving system (25), it can be shown that, for small δ, the following expressions hold

$$\begin{aligned} D_I(s) &= O(\sqrt{\delta}\ln\delta) \\ D_K(s) &= \frac{2\Omega\sqrt{\delta}[\Phi_1(s) - \Phi_2(s)]}{G[s\ln(R_2/R_1) + 4\Omega^2(R_1^{-2} - R_2^{-2})/s]} + O(\delta\ln\delta) \end{aligned} \tag{26}$$

substituting which into the right-hand side of Eq. (24) and using expansions of the cylindrical functions for small arguments, we obtain

$$U(r, s) = \frac{\Phi_1(s) - \Phi_2(s)}{r\rho(s^2 + \omega^2)\ln(R_2/R_1)} + O(\sqrt{\delta}\ln\delta), \quad \omega^2 = 2\frac{G(R_1^{-2} - R_2^{-2})}{\rho\ln(R_2/R_1)} \tag{27}$$

The expression for ω^2 agrees with the last formula of (17).

The result of inverting the principal (final) part of expression (27) is identical with the result which follows from formulae (11), (17) and (18).[7] Also taking into account the continuity of the function $u(r, t)$ with respect to each of the arguments, we arrive at the conclusion that by taking the limit $\nu \to 1/2 - 0$, the wave solution of boundary-value problem (22), (23) can be reduced to the solution (11), (17), (18), which it was required to prove.

As can be seen from relations (11), (17) and (18), external loads only occur in the formula for the displacements in the form of the difference $P_1(t) - P_2(t)$. Hence, the displacements and of course, also the strain tensor in the incompressible cylinder do not change, if, instead of the boundary pressures $P_1(t)$ and $P_2(t)$ we specify $P_1(t) + f(t)$ and $P_2(t) + f(t)$ respectively ($f(t)$ is any function which is the same for the inner and outer surfaces of the shell). This confirms the well-known fact that it is impossible to establish the stress tensor in an incompressible material uniquely, knowing only the displacements (or strains). For example, uniform compression of a hollow cylinder over the inner and outer surfaces simultaneously ($P_1(t) \equiv P_2(t)$) does not lead to displacements and strains, although the stress tensor will, naturally, not be zero. The stresses in an incompressible cylinder can only be determined uniquely if we know the boundary conditions, for example, using scheme (20) described above.

The convergence of the wave solutions for cylinders of slightly compressible materials to the analytical relations obtained (when condition (12) is satisfied and ν_{ij} approaches the values (7)), which holds on the assumption of incompressibility, can be clearly demonstrated using numerical methods of integrating the hyperbolic boundary-value problems. We considered a two-layer anisotropic cylinder with inner radius $R_1 = 0.1$ m, outer radius $R_2 = 0.12$ m and thicknesses of the layers $h_1 = h_2 = (R_2 - R_1)/2 = 0.01$ m. The layers were made of the same orthotropic composite material but with different reinforcing angles (the directions of the principal axes of anisotropy): for the inner layer $\alpha_1 = -\pi/6$ and for the outer layer $\alpha_2 = \pi/3$. The physical-mechanical characteristics of the composite material are as follows:[9]

$$E_x = E_r = 14 \cdot 10^3 \text{ MPa}, \quad E_\varphi = 56 \cdot 10^3 \text{ MPa}, \quad G_{x\varphi} = 5.7 \cdot 10^3 \text{ MPa}, \quad \rho = 2 \cdot 10^3 \text{ Kg/m}^3$$

In this case requirement (12) is satisfied for both layers.

The outer surface of the cylinder is load-free ($P_2(t) \equiv 0$), and the inner surface is loaded by a pressure impulse

$$P_1(t) = Q_0 e^{-t/T} H(t) \tag{28}$$

where $T = 10^{-5}$ s and $H(t)$ is the Heaviside function. A load of type (28) with a discontinuity at the front is a typical shock load, characteristic for a shock wave, which is formed when certain types of explosive materials explode.[3,4]

The coefficients of transverse strain ν_{ij} take the following values:

for an incompressible composite material, according to expressions (7), $\nu_{x\varphi} = 0.125$, $\nu_{\varphi r} = 0.5$ and $\nu_{rx} = 0.875$
for a slightly compressible composite material we assume $\nu_{x\varphi} = 0.12$, $\nu_{\varphi r} = 0.48$ and $\nu_{rx} = 0.87$;
for a composite material with a practical compressibility[9] $\nu_{x\varphi} = 0.07$, $\nu_{\varphi r} = 0.28$ and $\nu_{rx} = 0.4$.

For an incompressible cylinder for load (28) we obtain from formula (18)

$$Y(t) = \frac{Q_0 T^2}{M(1+\omega^2 T^2)}\left(e^{-t/T} - \cos\omega t + \frac{\sin\omega t}{\omega T}\right) \tag{29}$$

after which the remaining unknowns of the problem can also be easily written in terms of elementary functions using scheme (20), (13), (9). For a slightly compressible composite material and a composite material with a practical compressibility, the problem was solved numerically using a modified Wilkins algorithm.[10,11]

The results of analytic and numerical calculations are presented in Figs. 1–3. Curves 1 correspond to the analytic solution for an incompressible cylinder, curves 2 correspond to the numerical solution for a slightly compressible composite material, and curves 3 correspond to the numerical solution for a practical composite material. Fig. 1 demonstrates the change with time of the dimensionless displacement of the cylinder at the point of contact of the layers $r = (R_1 + R_2)/2$. In Fig. 2 we show the oscillations of the dimensionless stresses at an internal point of the inner layer $r = R_1 + 0$ and at an external point of the outer layer $r = R_2 - 0$. The disagreement between curves 1 and 2 in all the figures is small, which confirms that the wave solutions for a slightly compressed anisotropic cylinder converge to the analytic accurate solutions for an incompressible cylinder (when ν_{ij} converges to the values (7)). Here the convergence is uniform with respect to the displacement (and also with respect to $\tau_{x\varphi}$). The convergence with respect to the remaining stress components may be non-uniform: since the stresses are functions not only of u but also of $\partial u/\partial r$, naturally uniform convergence with respect to u will not always guarantee the same convergence with respect to $\partial u/\partial r$ and, as a consequence, with respect to the stresses. This applies particularly to the wave solutions with strong discontinuities, which will always occur in the case of shock loads of the type (28) with a discontinuity on the leading front.[12,13] Nevertheless, convergence with respect to the stresses on average is ensured, as can be seen from Figs. 2 and 3.

The function (29) when $t \approx 5.2 \times 10^{-5}$ s has a first local maximum (it is also global). In Fig. 3 we show the distribution of the circumferential stress over the thickness of the two-layer cylinder at this instant of time. As in Figs. 1 and 2, curves 1 and 2 agree well with one another. In a cylinder of practical composite, both the frequency and amplitude of the oscillations are slightly lower than in an incompressible cylinder, as indicated by curves 3 in all the figures.

The saw-tooth-shaped form of curves 2 in Fig. 2 for σ_φ and σ_x and in Fig. 3 is due to the fact that in a slightly compressible composite medium the propagation velocity of elastic waves under plane-strain conditions is a fairly high quantity, and hence after a calculated time a set of reflections of the input signal (28) both on the free surfaces from the interface of the layers occurs in the thickness of the two-layer cylinder. In the case of a shock load of the

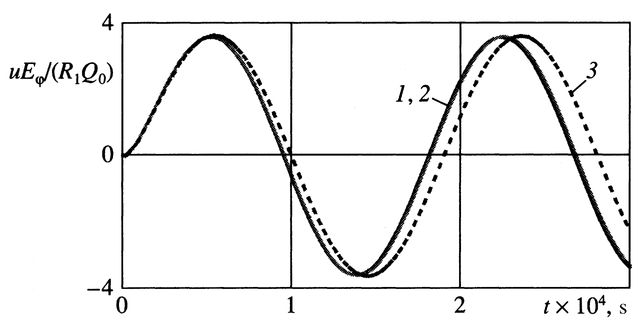

Fig. 1.

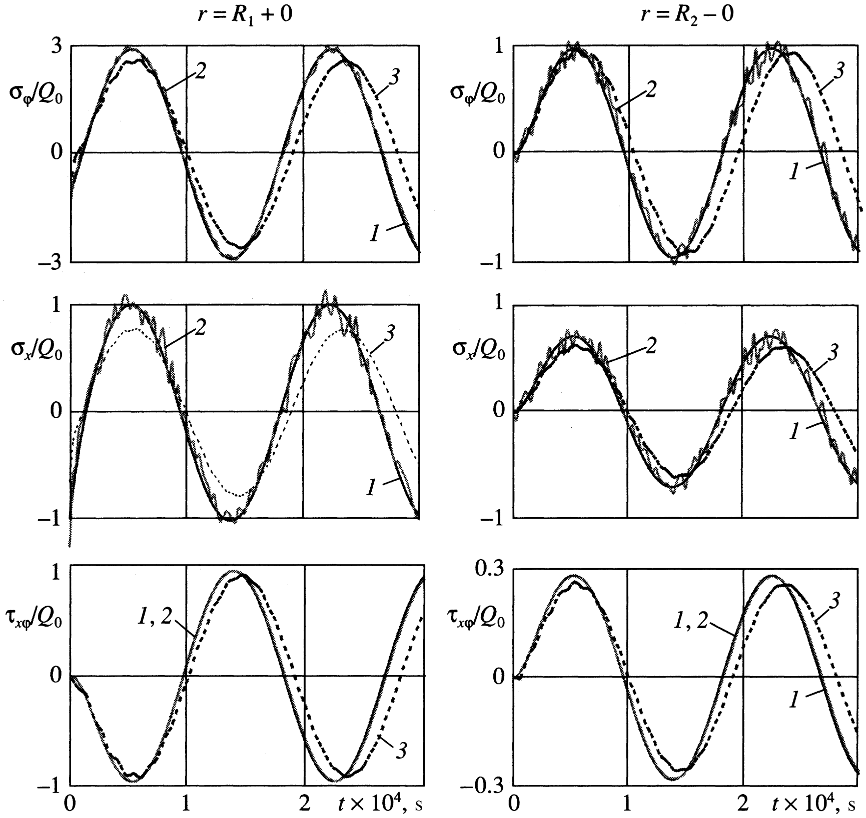

Fig. 2.

type (28) with a discontinuity on the leading edge, this will lead to a saw-tooth-shaped distribution of the stresses both in time and in the thickness of the shell.[12,13] In compressible materials the velocity of sound is considerably less, and this effect is much less pronounced (see curves 3 in Figs. 2 and 3). Moreover, curves 2 in all the figures were obtained by an approximate numerical method[11] of integrating the non-stationary hyperbolic boundary-value problems for cylinders made of compressible materials with finite velocities of sound. This method is not applicable for incompressible materials (under plane-strain conditions the velocities of sound in such media are infinite), while in the case of slightly compressible materials it may give an error in the form of the oscillations of the numerical solution around the exact solution. If the amplitude of the oscillations then exceeds 10% of the amplitude value of the

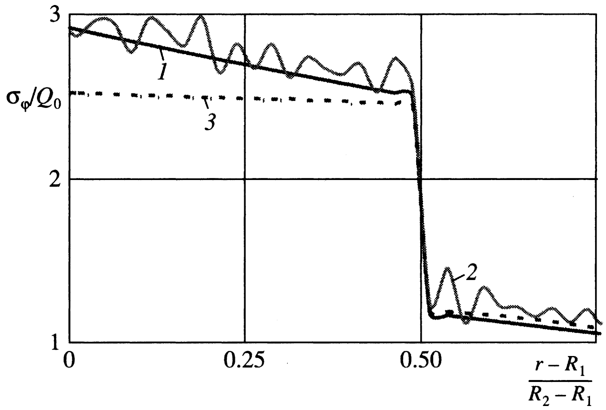

Fig. 3.

corresponding variable, the use of a numerical method[11] is not recommended. In the examples considered above, the level of the oscillations did not exceed 10% of the amplitude value of the corresponding stress, and, as can be seen in all the figures, the mean values of the numerical solutions (over a period of one oscillation) then turned out to be extremely close to the asymptotic accurate analytic solutions for an incompressible composite material (curves 2 oscillate slightly around curves 1). This enables us to assume that the numerical results obtained for a slightly compressible composite cylinder are fairly accurate (the error of calculations for the mean values over a period of one oscillation was no more than 3%, while the error of calculations for the amplitude values was less than 10%).

As the above results show, the analytic solutions (11), (17), (18), (20) enable us to give an accurate description of the one-dimensional dynamics of incompressible non-uniform helically orthotropic cylinders under plane-strain conditions and to give a fairly accurate description of slightly compressible cylinders. In the case of compressible materials, the proposed solutions (11), (17), (18), (20) can be used as a first approximation since they give only a slightly increased value of the frequency and amplitude values of the variables.

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